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Entangling two qubits by dissipation

L Jakóbczyk

Institute of Theoretical Physics, University of Wrocław, Pl. M Borna 9, 50-204 Wrocław, Poland

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Abstract
We discuss a model of dissipative dynamics of two qubits that can entangle some initially separable states.

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1. Introduction

As is well known, entangled states of composite quantum systems play a central role in quantum communication [1], quantum cryptography [2] and quantum computing [3]. Entanglement shows up when the system consists of two or more subsystems and the total state cannot be written as a product state. Pure entangled states are superpositions of multiparticle states and as such are extremely fragile with respect to noise. In practical realization, every quantum system is open, and unavoidable interaction with its environment results in dissipation and destruction of correlations. As a consequence, even if initially some amount of entanglement is present in the system, it will subsequently disappear. Recently, however, the interesting idea that dissipation can create rather than destroy entanglement in some systems, was put forward in several publications [4–7]. This possibility is also considered in the present paper. We study the simplest composite quantum system consisting of two qubits (two-level quantum subsystems). The qubits are coupled to a common thermostat at zero temperature and the reduced dynamics (in the Markovian approximation) is given by the semi-group $\{T_t\}$ of completely positive linear mappings acting on density matrices [8]. When qubits are realized by two-level atoms, this kind of dynamics takes into account only spontaneous emission and possible photon exchange between atoms [9]. In that case, the generator L of $\{T_t\}$ is parametrized in terms of the spontaneous emission rate of the single atom γ_0 and the photon exchange rate γ. In the following, we focus on the realization of qubits by two-level atoms and consider time evolution of the initial state of the system as well as the evolution of its entanglement, measured by the so-called concurrence [10, 11]. First, we study the mathematically idealized situation when the photon exchange rate γ is close to γ_0 and we can use the approximation $\gamma = \gamma_0$ (a model with a similar choice of γ was also considered in [12]). The approximation is legitimate when two atoms are separated by a small distance compared to the radiation wavelength. In such a case there is a substantial probability that a photon emitted by one atom will be absorbed by the other and the photon exchange process can produce correlations between atoms which partially overcome decoherence caused by spontaneous emission. We analytically investigate the properties of the semi-group generated by L with $\gamma = \gamma_0$ and we calculate its asymptotic stationary states ρ_{as}. The results show that they depend on initial conditions (i.e. the semi-group is relaxing but not uniquely relaxing). The concurrence of ρ_{as} also depends on the initial state and can be nonzero for some of them. We discuss in detail some classes of initial states. In particular, we show that there are pure separable states evolving to entangled mixed states and which remain separable during evolution. The class of pure maximally entangled initial states is also discussed. Similar 'production' of entanglement is shown to be present for some classes of mixed states. On the other hand, for the more realistic model in which the photon exchange rate is smaller than γ_0 and depends on the interatomic distance, the relaxation process brings all initial states to the unique asymptotic state when both the atoms are in their ground states. Even in that case the dynamics can entangle two separable states, but the amount of entanglement decreases to zero.

It is worth stressing that the paper discusses the theoretical possibility of the creation of entanglement by a purely incoherent dissipative process. We show that this is possible in principle; depending on the initial state the dynamics can create rather than destroy entanglement. However, experimental realization of such a process with real two-level atoms seems to be problematic, mainly due to the difficulty in preparation of relevant initial states. For example, preparation of the state when one atom is in an excited state and the other in the ground state, which evolves to an asymptotic state with nonzero entanglement, may not be possible when the atoms are separated by a distance less than radiation wavelength. On the other hand, when the distance is larger, the timescale on which the produced entanglement is maximal may be small. Similarly, the detection of entanglement by some physical measurement process is difficult. It is impossible during the time evolution of the system without destroying correlations leading to entanglement. For asymptotic states, one can try to detect the violation of some CHSH-Bell inequality. But it may not work for mixed states [13, 14].

2. Pairs of two-level atoms

Consider a two-level atom A with ground state $|0\rangle$ and excited state $|1\rangle$. This quantum system can be described in terms of the Hilbert space $\mathcal{H}_A = \mathbb{C}^2$ and the algebra $\mathfrak{A}_A$ of 2×2 complex matrices. If we identify $|1\rangle$ and $|0\rangle$ with vectors $\binom{1}{0}$ and $\binom{0}{1}$ respectively, then the raising and lowering operators σ_+, σ_- defined by

$$\sigma_+ = |1\rangle\langle 0| \qquad \sigma_- = |0\rangle\langle 1| \tag{1}$$

can be expressed in terms of Pauli matrices σ_1, σ_2

$$\sigma_+ = \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) \qquad \sigma_- = \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2). \tag{2}$$

For a joint system AB of two two-level atoms A and B, the algebra $\mathfrak{A}_{AB}$ is equal to 4×4 complex matrices and the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^4$. Let $\mathcal{E}_{AB}$ be the set of all states of the compound system, i.e.

$$\mathcal{E}_{AB} = \{\rho \in \mathfrak{A}_{AB} : \rho \geqslant 0 \text{ and } \operatorname{tr}\rho = 1\}. \tag{3}$$

The state $\rho \in \mathcal{E}_{AB}$ is *separable* [13], if it has the form

$$\rho = \sum_k \lambda_k \rho_k^A \otimes \rho_k^B \qquad \rho_k^A \in \mathcal{E}_A \qquad \rho_k^B \in \mathcal{E}_B, \lambda_k \geqslant 0 \qquad \text{and} \qquad \sum_k \lambda_k = 1. \tag{4}$$

The set $\mathcal{E}_{AB}^{\mathrm{sep}}$ of all separable states forms a convex subset of $\mathcal{E}_{AB}$. When ρ is not separable, it is called *inseparable* or *entangled*. Thus

$$\mathcal{E}_{AB}^{\mathrm{ent}} = \mathcal{E}_{AB} \setminus \mathcal{E}_{AB}^{\mathrm{sep}}. \tag{5}$$

If $P \in \mathcal{E}_{AB}$ is a pure state, i.e. P is a one-dimensional projector, then P is separable iff partial traces $\mathrm{tr}_A P$ and $\mathrm{tr}_B P$ are also projectors. For mixed states, the separability problem is much more involved. Fortunately, in the case of a four-level compound system there is a basic necessary and sufficient condition for separability: ρ is separable iff its partial transposition ρ^{T_A} is also a state [15]. Another interesting question is how to measure the amount of entanglement a given quantum state contains. For a pure state P, the entropy of entanglement

$$E(P) = -\mathrm{tr}\,[(\mathrm{tr}_A P)\log_2(\mathrm{tr}_A P)] \tag{6}$$

is essentially a unique measure of entanglement [16]. For a mixed state ρ, it seems that the basic measure of entanglement is the entanglement of formation [17]

$$E(\rho) = \min \sum_k \lambda_k E(P_k) \tag{7}$$

where the minimum is taken over all possible decompositions

$$\rho = \sum_k \lambda_k P_k. \tag{8}$$

Again, in the case of a four-level system, $E(\rho)$ can be explicitly computed and it turns out that $E(\rho)$ is the function of another useful quantity $C(\rho)$ called *concurrence*, which also can be taken as a measure of entanglement [10, 11]. Since in this paper we use concurrence to quantify entanglement, we now discuss its definition. Let

$$\rho^\dagger = (\sigma_2 \otimes \sigma_2)\,\overline{\rho}\,(\sigma_2 \otimes \sigma_2) \tag{9}$$

where $\overline{\rho}$ is the complex conjugation of the matrix ρ. Also define

$$\widehat{\rho} = (\rho^{1/2}\rho^\dagger\rho^{1/2})^{1/2}. \tag{10}$$

Then the concurrence $C(\rho)$ is given by [10, 11]

$$C(\rho) = \max(0, 2p_{\max}(\widehat{\rho}) - \mathrm{tr}\,\widehat{\rho}) \tag{11}$$

where $p_{\max}(\widehat{\rho})$ denotes the maximal eigenvalue of $\widehat{\rho}$. The value of the number $C(\rho)$ varies from 0 for separable states, to 1 for maximally entangled pure states.

3. Decay in a system of closely separated atoms

We study the spontaneous emission of two atoms separated by a distance R, small compared to the radiation wavelength. At such distances there is a substantial probability that the photon emitted by one atom will be absorbed by the other. Thus the dynamics of the system is given by the master equation [9]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = L\rho \qquad \rho \in \mathcal{E}_{AB} \tag{12}$$

with the following generator L

$$\begin{aligned} L\rho = \frac{\gamma_0}{2}&\left[2\sigma_-^A\,\rho\sigma_+^A + 2\sigma_-^B\,\rho\sigma_+^B - \left(\sigma_+^A\sigma_-^A + \sigma_+^B\sigma_-^B\right)\rho - \rho\left(\sigma_+^A\sigma_-^A + \sigma_+^B\sigma_-^B\right)\right] \\ &+ \frac{\gamma}{2}\left[2\sigma_-^A\,\rho\sigma_+^B + 2\sigma_-^B\,\rho\sigma_+^A - \left(\sigma_+^A\sigma_-^B + \sigma_+^B\sigma_-^A\right)\rho - \rho\left(\sigma_+^A\sigma_-^B + \sigma_+^B\sigma_-^A\right)\right] \end{aligned} \tag{13}$$

where

$$\sigma_\pm^A = \sigma_\pm \otimes \mathbb{I} \qquad \sigma_\pm^B = \mathbb{I} \otimes \sigma_\pm \qquad \sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2). \tag{14}$$

Here γ_0 is the single atom spontaneous emission rate, and $\gamma = g\gamma_0$ is a relaxation constant of photon exchange. In the model, g is a function of the distance R between atoms, and $g \to 1$ when $R \to 0$. In this section we investigate the time evolution of the initial density matrix ρ of the compound system, governed by the semi-group $\{T_t\}_{t\geqslant 0}$ generated by L. In particular, we will study the time development of entanglement of ρ, measured by concurrence.

Assume that the distance between atoms is so small that the exchange rate γ is close to γ_0 and we can use the approximation $g = 1$. Under this condition we study evolution of the system and in particular we consider asymptotic states. Direct calculations show that the semi-group $\{T_t\}$ generated by L with $g = 1$ is relaxing but not uniquely relaxing, i.e. there are as many stationary states as there are initial conditions. More precisely, for a given initial state $\rho = (\rho_{jk})$, the state $\rho(t)$ at time t has the following matrix elements with respect to the basis $e_1 = |1\rangle \otimes |1\rangle$, $e_2 = |1\rangle \otimes |0\rangle$, $e_3 = |0\rangle \otimes |1\rangle$, $e_4 = |0\rangle \otimes |0\rangle$

$$
\begin{aligned}
\rho_{11}(t) &= \mathrm{e}^{-2\gamma_0 t}\rho_{11}\\
\rho_{12}(t) &= \tfrac{1}{2}[\mathrm{e}^{-2\gamma_0 t}(\rho_{12}+\rho_{13}) + \mathrm{e}^{-\gamma_0 t}(\rho_{12}-\rho_{13})]\\
\rho_{13}(t) &= \tfrac{1}{2}[\mathrm{e}^{-2\gamma_0 t}(\rho_{12}+\rho_{13}) + \mathrm{e}^{-\gamma_0 t}(\rho_{13}-\rho_{12})]\\
\rho_{14}(t) &= \mathrm{e}^{-\gamma_0 t}\rho_{14}\\
\rho_{22}(t) &= \tfrac{1}{4}\,\mathrm{e}^{-2\gamma_0 t}(\rho_{22}+\rho_{33}+2\mathrm{Re}\,\rho_{23}) + \tfrac{1}{2}\,\mathrm{e}^{-\gamma_0 t}(\rho_{22}-\rho_{33}) + \gamma_0 t\,\mathrm{e}^{-2\gamma_0 t}\rho_{11}\\
&\quad + \tfrac{1}{4}(\rho_{22}+\rho_{33}-2\mathrm{Re}\,\rho_{23})\\
\rho_{23}(t) &= \tfrac{1}{4}\,\mathrm{e}^{-2\gamma_0 t}(\rho_{22}+\rho_{33}+2\mathrm{Re}\,\rho_{23}) + \tfrac{1}{2}\,\mathrm{e}^{-\gamma_0 t}(\rho_{23}-\rho_{32}) + \gamma_0 t\,\mathrm{e}^{-2\gamma_0 t}\rho_{11}\\
&\quad - \tfrac{1}{4}(\rho_{22}+\rho_{33}-2\mathrm{Re}\,\rho_{23})\\
\rho_{24}(t) &= -\tfrac{1}{2}\,\mathrm{e}^{-2\gamma_0 t}(\rho_{12}+\rho_{13}) + \tfrac{1}{2}\,\mathrm{e}^{-\gamma_0 t}(2\rho_{12}+2\rho_{13}+\rho_{24}+\rho_{34}) + \tfrac{1}{2}(\rho_{24}-\rho_{34})\\
\rho_{33}(t) &= \rho_{22}(t)\\
\rho_{34}(t) &= -\tfrac{1}{2}\,\mathrm{e}^{-2\gamma_0 t}(\rho_{12}+\rho_{13}) + \tfrac{1}{2}\,\mathrm{e}^{-\gamma_0 t}(2\rho_{12}+2\rho_{13}+\rho_{24}+\rho_{34}) - \tfrac{1}{2}(\rho_{24}-\rho_{34})\\
\rho_{44}(t) &= -\mathrm{e}^{-2\gamma_0 t}(\rho_{11}+\rho_{22}+\mathrm{Re}\,\rho_{23}) - 2\gamma_0 t\,\mathrm{e}^{-2\gamma_0 t}\rho_{11} + \tfrac{1}{2}(1+\rho_{11}+\rho_{44}+2\mathrm{Re}\,\rho_{23})
\end{aligned}
$$

and the remaining matrix elements can be obtained by the Hermiticity condition $\rho_{kj} = \overline{\rho}_{jk}$. In the limit $t \to \infty$ we obtain asymptotic (stationary) states parametrized as follows:

$$
\rho_{\mathrm{as}} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & \alpha & -\alpha & \beta\\ 0 & -\alpha & \alpha & -\beta\\ 0 & \overline{\beta} & -\overline{\beta} & 1-2\alpha \end{pmatrix} \tag{15}
$$

where

$$
\alpha = \tfrac{1}{4}(\rho_{22}+\rho_{33}-2\mathrm{Re}\,\rho_{23}) \qquad \beta = \tfrac{1}{2}(\rho_{24}-\rho_{34}). \tag{16}
$$

We can also compute the concurrence of the asymptotic state and the result is: *the concurrence of asymptotic state of the semi-group $\{T_t\}$ generated by L with $g = 1$* is *equal to*

$$
C(\rho_{\mathrm{as}}) = 2|\alpha| = \tfrac{1}{2}|\rho_{22}+\rho_{33}-2\mathrm{Re}\,\rho_{23}| \tag{17}
$$

where ρ_{jk} are the matrix elements of the initial state.

4. Some examples

In this section we consider examples of initial states and their evolution.

4.1. Pure separable states

Let

$$\rho = P_{\Psi\otimes\Phi} = P_{\Psi} \otimes P_{\Phi} \tag{18}$$

where

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \in \mathcal{H}_A \qquad \Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} \in \mathcal{H}_B$$

are normalized. Then one can find out that

$$\alpha = \tfrac{1}{4}(1 - |\langle\Psi, \Phi\rangle|^2) \qquad \beta = \tfrac{1}{2}(|\Phi_2|^2\Psi_1\overline{\Psi}_2 - |\Psi_2|^2\Phi_1\overline{\Phi}_2) \tag{19}$$

where $\langle\cdot, \cdot\rangle$ is the inner product in $\mathbb{C}^2$. So

$$C(\rho_{\mathrm{as}}) = \tfrac{1}{2}(1 - |\langle\Psi, \Phi\rangle|^2). \tag{20}$$

From formula (21) we see that there are separable initial states for which asymptotic states are entangled. In particular, the asymptotic state has a maximal concurrence if vectors Ψ and Φ are orthogonal and their concurrence is zero (the state remains separable) if $|\langle\Psi, \Phi\rangle| = 1$.

Now we discuss some special cases.

(a) When one atom is in an excited state and the other in the ground state

$$\Psi = |1\rangle \qquad \Phi = |0\rangle$$

the asymptotic (mixed) state is given by

$$\rho_{\mathrm{as}} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & \frac{1}{4} & -\frac{1}{4} & 0\\ 0 & -\frac{1}{4} & \frac{1}{4} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can also be shown that in this case the relaxation process produces the state ρ_t with concurrence

$$C(\rho_t) = \frac{1 - \mathrm{e}^{-\gamma_0 t}}{2}$$

increasing to the maximal value equal to 1/2. Thus two atoms initially in separable states become entangled for all t and the asymptotic (steady) state attains the maximal amount of entanglement.

(b) When two atoms are in excited states

$$\Psi = \Phi = |1\rangle$$

the asymptotic state is equal to

$$|0\rangle \otimes |0\rangle.$$

Thus the relaxation process brings the two atoms into the ground state.

(c) The $|0\rangle \otimes |0\rangle$ state is stationary for the semi-group $\{T_t\}$.

4.2. Pure maximally entangled states

Let

$$\rho = Q(a,\theta_1,\theta_2) = \begin{pmatrix} \frac{a^2}{2} & \frac{a\sqrt{1-a^2}}{2}\,\mathrm{e}^{-\mathrm{i}\theta_1} & \frac{a\sqrt{1-a^2}}{2}\,\mathrm{e}^{-\mathrm{i}\theta_2} & -\frac{a^2}{2}\,\mathrm{e}^{-\mathrm{i}(\theta_1+\theta_2)} \\ \frac{a\sqrt{1-a^2}}{2}\,\mathrm{e}^{\mathrm{i}\theta_1} & \frac{1-a^2}{2} & \frac{1-a^2}{2}\,\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} & -\frac{a\sqrt{1-a^2}}{2}\,\mathrm{e}^{-\mathrm{i}\theta_2} \\ \frac{a\sqrt{1-a^2}}{2}\,\mathrm{e}^{\mathrm{i}\theta_2} & \frac{1-a^2}{2}\,\mathrm{e}^{-\mathrm{i}(\theta_1-\theta_2)} & \frac{1-a^2}{2} & -\frac{a\sqrt{1-a^2}}{2}\,\mathrm{e}^{-\mathrm{i}\theta_1} \\ -\frac{a^2}{2}\,\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)} & -\frac{\sqrt{1-a^2}}{2}\,\mathrm{e}^{\mathrm{i}\theta_2} & -\frac{a\sqrt{1-a^2}}{2}\,\mathrm{e}^{\mathrm{i}\theta_1} & \frac{a^2}{2} \end{pmatrix}$$

where $a \in [0,1]$, $\theta_1, \theta_2 \in [0, 2\pi]$. Pure states $Q(a,\theta_1,\theta_2)$ are maximally entangled and form a family of all maximally entangled states of the four-level system [19]. It turns out that ρ_{as} is defined by

$$\begin{aligned} \alpha &= \tfrac{1}{4}(1-a^2)(1-\cos(\theta_1-\theta_2)) \\ \beta &= \tfrac{1}{4}a\sqrt{1-a^2}(\mathrm{e}^{-\mathrm{i}\theta_1} - \mathrm{e}^{-\mathrm{i}\theta_2}) \end{aligned} \tag{21}$$

and

$$C(\rho_{\mathrm{as}}) = \tfrac{1}{2}(1-a^2)(1-\cos(\theta_1-\theta_2)). \tag{22}$$

From formula (23) we see that there are initial maximally entangled states which asymptotically become separable ($a = 1$ or $\theta_1 - \theta_2 = 2k\pi$) such that the asymptotic concurrence is greater than 0. States with $a = 0$ and $\theta_1 - \theta_2 = (2k+1)\pi$ remain maximally entangled. For example the state

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \tag{23}$$

is stable. On the other hand, the concurrence of

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \tag{24}$$

goes to zero faster than the concurrence of

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \tag{25}$$

as shown in figure 1. In Dicke's theory of spontaneous radiation processes the state (24) is called subradiant whereas the state (25) has half the lifetime of a single atom and therefore is called superradiant [18]. We see that the time-dependence of concurrence reflects the relaxation properties of those states.

4.3. Some classes of mixed states

(a) Bell-diagonal states. Let

$$\rho_{\mathrm{B}} = p_1|\Phi^+\rangle\langle\Phi^+| + p_2|\Phi^-\rangle\langle\Phi^-| + p_3|\Psi^+\rangle\langle\Psi^+| + p_4|\Psi^-\rangle\langle\Psi^-| \tag{26}$$

where Bell states $\Phi^\pm$ and $\Psi^\pm$ are given by

$$\Phi^\pm = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle) \qquad \Psi^\pm = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle \pm |0\rangle \otimes |1\rangle) \tag{27}$$

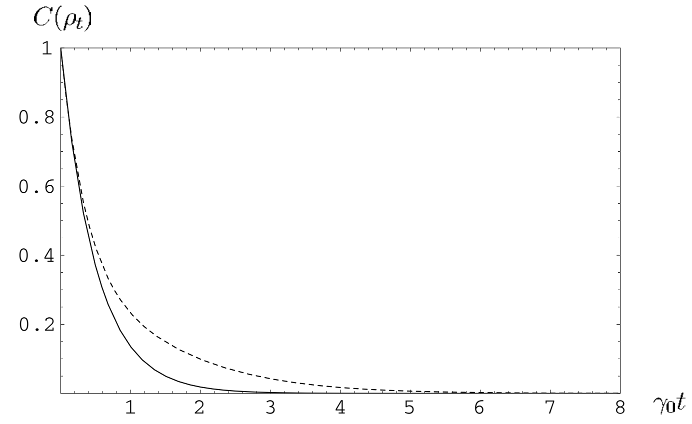

Figure 1. Concurrence as a function of time for initial states: $\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$ (dotted line) and $\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$ (solid line).

It is known that all $p_i \in [0, 1/2]$, ρ_{B} is separable, while for $p_1 > 1/2$, ρ_{B} is entangled with concurrence equal to $2p_1 - 1$ (similarly for p_2, p_3, p_4) [20]. Now the asymptotic state has the form

$$\rho_{\mathrm{as}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{p_4}{2} & -\frac{p_4}{2} & 0 \\ 0 & -\frac{p_4}{2} & \frac{p_4}{2} & 0 \\ 0 & 0 & 0 & 1-p_4 \end{pmatrix} \tag{28}$$

with concurrence $C(\rho_{\mathrm{as}}) = p_4$. So even when the initial state is separable, the asymptotic state becomes entangled.

(b) Werner states [21]. Let

$$\rho_{\mathrm{W}} = (1-p)\frac{\mathbb{I}_4}{4} + p|\Phi^+\rangle\langle\Phi^+|. \tag{29}$$

If $p > 1/3$, ρ_{W} is entangled with concurrence equal to $(3p-1)/2$. However

$$\rho_{\mathrm{as}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1-p}{8} & \frac{p-1}{8} & 0 \\ 0 & \frac{p-1}{8} & \frac{1-p}{8} & 0 \\ 0 & 0 & 0 & \frac{3+p}{4} \end{pmatrix} \tag{30}$$

has the concurrence $C(\rho_{\mathrm{as}}) = \frac{1-p}{4}$, so the asymptotic states are entangled for all $p \neq 1$. Note that even completely mixed state $\frac{\mathbb{I}_4}{4}$ evolves to an entangled asymptotic state.

(c) Maximally entangled mixed states. The states

$$\rho_{\mathrm{M}} = \begin{pmatrix} h(\delta) & 0 & 0 & \delta/2 \\ 0 & 1-2h(\delta) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta/2 & 0 & 0 & h(\delta) \end{pmatrix} \qquad h(\delta) = \begin{cases} 1/3 & \delta \in [0, 2/3] \\ \delta/2 & \delta \in [2/3, 1] \end{cases} \tag{31}$$

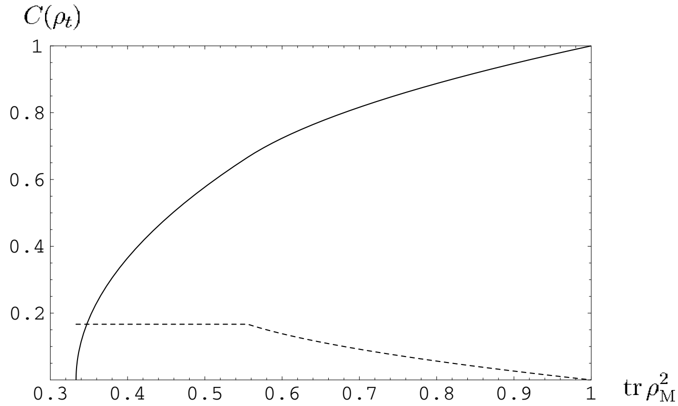

Figure 2. Concurrence of ρ_{M} (solid line) and ρ_{as} (dotted line) as a function of tr ρ_{M}^2.

are conjectured to be maximally entangled for a given degree of impurity measured by tr ρ^2 [22]. According to (18) the concurrence of the asymptotic state is given by

$$C(\rho_{\mathrm{as}}) = \tfrac{1}{2}(1 - 2h(\delta)) \tag{32}$$

Even in that case, there are initial states (for sufficiently small tr ρ_{M}^2) such that the asymptotic state is more entangled (see figure 2).

5. Remarks on the general case

In the case of arbitrary distance between the atoms, i.e. when $g \in [0, 1)$, the semi-group generated by L is uniquely relaxing, with the asymptotic state $|0\rangle \otimes |0\rangle$. Thus, for any initial state ρ, the concurrence $C(\rho_t)$ approaches 0 when $t \to \infty$. But it does not mean that the function $t \to C(\rho_t)$ is always monotonic. The general form of $C(\rho_t)$ is rather involved, so we consider only some special cases.

(1) Let the initial state of the compound system be equal to $|0\rangle \otimes |1\rangle$. This state evolves to

$$\rho_t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\,\mathrm{e}^{-\gamma_0 t}(\cosh\gamma t + 1) & -\frac{1}{2}\,\mathrm{e}^{-\gamma_0 t}\sinh\gamma t & 0 \\ 0 & -\frac{1}{2}\,\mathrm{e}^{-\gamma_0 t}\sinh\gamma t & \frac{1}{2}\,\mathrm{e}^{-\gamma_0 t}(\cosh\gamma t - 1) & 0 \\ 0 & 0 & 0 & 1 - \mathrm{e}^{-\gamma_0 t}\cosh\gamma t \end{pmatrix} \tag{33}$$

with concurrence

$$C(\rho_t) = \mathrm{e}^{-\gamma_0 t}\sinh\gamma t. \tag{34}$$

In the interval $[0, t_\gamma]$, where

$$t_\gamma = \frac{1}{2\gamma}\ln\frac{\gamma_0 + \gamma}{\gamma_0 - \gamma}$$

the function (34) is increasing to its maximal value

$$C_{\max} = \frac{\gamma}{\gamma_0 - \gamma}\left(\frac{\gamma_0 + \gamma}{\gamma_0 - \gamma}\right)^{-\frac{\gamma_0+\gamma}{2\gamma}}$$

whereas for $t > t_\gamma$, $C(\rho_t)$ decreases to 0. Thus for any nonzero photon exchange rate γ, dynamics given by the semi-group $\{T_t\}$ produces some amount of entanglement between two

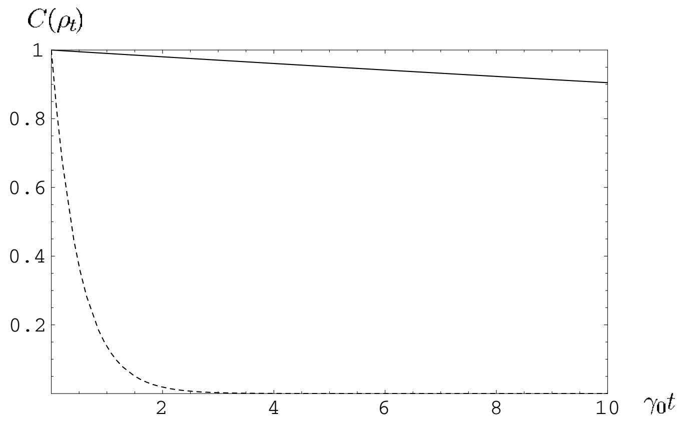

Figure 3. $C(\rho_t^+)$ (dotted line) and $C(\rho_t^-)$ (solid line) for $\gamma/\gamma_0 = 0.99$.

atoms which are initially in the ground state and excited state. Note that the maximal value of $C(\rho_t)$ depends only on emission rates γ_0 and γ.

(2) For the initial states

$$\Psi^{\pm} = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle)$$

the relaxation to the asymptotic state $|0\rangle \otimes |0\rangle$ is given by density matrices

$$\rho_t^{\pm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\,\mathrm{e}^{-(\gamma_0 \pm \gamma)t} & -\frac{1}{2}\,\mathrm{e}^{-(\gamma_0 \pm \gamma)t} & 0 \\ 0 & -\frac{1}{2}\,\mathrm{e}^{-(\gamma_0 \pm \gamma)t)} & \frac{1}{2}\,\mathrm{e}^{-(\gamma_0 \pm \gamma)t} & 0 \\ 0 & 0 & 0 & 1 - \mathrm{e}^{-(\gamma_0 \pm \gamma)t} \end{pmatrix} \tag{35}$$

with the corresponding concurrence

$$C\left(\rho_t^{\pm}\right) = \mathrm{e}^{-(\gamma_0 \pm \gamma)t}.$$

The state Ψ^- is no longer stable (as in the case of $\gamma = \gamma_0$), but during the evolution its concurrence goes to zero slower than $C\left(\rho_t^+\right)$ (figure 3). For γ close to γ_0, Ψ^- is almost stable.

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